

Fast Recovery and Approximation of Hidden Cauchy Structure

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Abstract

We derive an algorithm of optimal complexity which determines whether a given matrix is a Cauchy matrix, and which exactly recovers the Cauchy points defining a Cauchy matrix from the matrix entries. Moreover, we study how to approximate a given matrix by a Cauchy matrix with a particular focus on the recovery of Cauchy points from noisy data. We derive an approximation algorithm of optimal complexity for this task, and prove approximation bounds. Numerical examples illustrate our theoretical results.

1 Introduction

Two vectors $s \in \mathbb{C}^m$, $t \in \mathbb{C}^n$ are called *Cauchy points*, if

$$s_i - t_j \neq 0 \quad \text{for all } i, j.$$

Such Cauchy points define a *Cauchy matrix*

$$C(s, t) = [c_{ij}] := \left[\frac{1}{s_i - t_j} \right].$$

Cauchy matrices occur in numerous applications. To give just one example, let $(s_i, z_i) \in \mathbb{C} \times \mathbb{C}$ be given with pairwise distinct values s_1, \dots, s_n and let $t_1, \dots, t_n \in \mathbb{C}$ be given with $s_i \neq t_j$ for all i, j . Then the coefficients $a = [a_1, \dots, a_n]^T \in \mathbb{C}^n$ such that the rational function

$$r(\zeta) = \sum_{j=1}^n \frac{a_j}{\zeta - t_j}$$

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satisfies $r(s_i) = z_i$, $i = 1, \dots, n$, can be found by solving the linear system

$$C(s, t)a = z.$$

Note that the condition $s_i - t_j \neq 0$ for the Cauchy points appears naturally in this application (as in many others) by the requirement that the poles of the rational function $r(\zeta)$ must be distinct from the points where the (finite) values of $r(\zeta)$ are prescribed.

A Cauchy matrix satisfies the Sylvester type displacement equation

$$SC(s, t) - C(s, t)T = 1_m 1_n^T,$$

where $S := \text{diag}(s) \in \mathbb{C}^{m,m}$, $T := \text{diag}(t) \in \mathbb{C}^{n,n}$, and $1_m := [1, \dots, 1]^T \in \mathbb{R}^m$. Hence the $\{S, T\}$ -displacement rank of $C(s, t)$ is equal to 1. The concept of displacement rank was originally introduced in [2, 9]; see [5, Section 12.1] for an introduction. Due to this special structure, several fast algorithms exist for performing matrix computations with $C(s, t)$. For example, an LU decomposition of $C(s, t)$ with partial pivoting can be computed in $\mathcal{O}(mn)$ operations [3] (the GKO algorithm), and matrix-vector products with $C(s, t)$ can be computed very fast [6] (the fast multipole method); see also [4] and [10, Section 3.6].

In this work we are, however, not concerned with performing computations with Cauchy matrices. Rather we study the problem of determining whether a given matrix $A \in \mathbb{C}^{m,n}$ is equal or at least “close” to a Cauchy matrix. For such matrices we derive algorithms of *optimal complexity* that compute Cauchy points $s \in \mathbb{C}^m$, $t \in \mathbb{C}^n$ with $A = C(s, t)$ when A is a Cauchy matrix, or with $A \approx C(s, t)$ when certain conditions are satisfied. We are not aware that a similar study has appeared in the literature before.

This cheap recognition (and approximation) could possibly be useful in black-box linear system solvers: Instead of using a general purpose method, one could first run the proposed algorithms in order to determine whether the given matrix is close to a Cauchy matrix, and then solve the system with a specialized algorithm. The upfront test runs in time proportional to the size of the input, and hence the computational overhead is negligible.

Let us briefly describe our general approach and the outline of this paper. When $A = [a_{ij}] = C(s, t)$ is a Cauchy matrix, but the corresponding Cauchy points s, t are unknown, these can be computed by solving the mn nonlinear equations (in $m + n$ variables)

$$\frac{1}{s_i - t_j} = a_{ij}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n. \quad (1)$$

For the Cauchy matrix A we have $a_{ij} \neq 0$, and hence the equations (1) are *equivalent* to the nm linear equations (in $m + n$ variables)

$$s_i - t_j = \frac{1}{a_{ij}}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n. \quad (2)$$

In Section 2 we discuss the *linearization* (2) of the equations (1) in more detail, study uniqueness properties of its solution and derive an algorithm for solving (2) in $\mathcal{O}(m + n)$ operations.

If the given matrix $A = [a_{ij}]$ is not a Cauchy matrix, and the task is to *approximate* A with a Cauchy matrix, one would ideally like to solve the nonlinear optimization problem

$$\min_{s,t} \sum_{i=1}^m \sum_{j=1}^n \left| \frac{1}{s_i - t_j} - a_{ij} \right|^2 = \min_{s,t} \|C(s,t) - A\|_F^2. \quad (3)$$

Instead of solving (3), we consider the linear least squares problem

$$\min_{s,t} \sum_{i=1}^m \sum_{j=1}^n \left| s_i - t_j - \frac{1}{a_{ij}} \right|^2 = \min_{s,t} \|D(s,t) - A^{[-1]}\|_F^2, \quad (4)$$

where

$$A^{[-1]} := [a_{ij}^{-1}], \quad D(s,t) := [s_i - t_j] \in \mathbb{C}^{m,n}.$$

The problem (4) can be considered a *linearization* of the nonlinear problem (3). We first show in Section 3.1 how to solve (4) in $\mathcal{O}(nm)$ operations. In Section 3.2 we relate the solutions obtained from (4) to solutions of the original problem (3). In particular, we analyze when a solution of (4) delivers a good approximation to the Cauchy points of a “noisy” Cauchy matrix $A = C(s,t) + N$, where the matrix N represents some data error. We illustrate our results by numerical experiments in Section 3.3. Concluding remarks are given in Section 4.

Notation The vector (matrix) of all ones in \mathbb{C}^n ($\mathbb{C}^{m,n}$) is denoted by 1_n ($1_{m,n}$). For a matrix $A = [a_{ij}] \in \mathbb{C}^{m,n}$,

$$\|A\|_F = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad \|A\|_M = \max_{i,j} |a_{ij}|$$

denote its Frobenius and maximum norm, respectively. Provided that all the entries of A are nonzero, its elementwise inverse is $A^{[-1]} := [a_{ij}^{-1}]$, and $A^{[-T]} := (A^{[-1]})^T$. For two matrices A, B of appropriate sizes we denote by $A \odot B$ and $A \otimes B$ their Hadamard (elementwise) and Kronecker products, respectively. Finally, $\text{vec}(A) \in \mathbb{C}^{mn}$ denotes the vector resulting from stacking all the columns of $A \in \mathbb{C}^{m,n}$ upon another.

2 Exact recovery of Cauchy points

Let $A = [a_{ij}] \in \mathbb{C}^{m,n}$ with $a_{ij} \neq 0$ for all i, j be given. There exist Cauchy points $s \in \mathbb{C}^m$, $t \in \mathbb{C}^n$ with $A = C(s,t)$, i.e., A is a Cauchy matrix, if and only if the equations (1) hold. Since $a_{ij} \neq 0$ for all i, j , the equations (1) are equivalent with the equations (2), and these can be written in matrix form as

$$U \begin{bmatrix} s \\ t \end{bmatrix} = b, \quad (5)$$

where

$$U := [I_m \otimes 1_n \quad -1_m \otimes I_n] \in \mathbb{C}^{mn, (m+n)}, \quad b := \text{vec}(A^{[-T]}) \in \mathbb{C}^{mn}. \quad (6)$$

Using the (overdetermined) linear system (5)–(6) we can test whether a given matrix $A = [a_{ij}]$ with $a_{ij} \neq 0$ for all i, j is a Cauchy matrix or not:

If $\begin{bmatrix} s \\ t \end{bmatrix}$ solves (5)–(6) for a componentwise nonzero right hand side b , then $s_i - t_j \neq 0$ for all i, j (cf. (2)), so that s, t are Cauchy points and $A = C(s, t)$. On the other hand, there are, of course, matrices A with all entries nonzero, giving a componentwise nonzero b , for which no solution of (5)–(6) exists.

Example 2.1. For $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ we have

$$U = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad b = \text{vec}(A^{[-T]}) = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix},$$

and a simple computation shows that there exists no solution of $U \begin{bmatrix} s \\ t \end{bmatrix} = b$. Hence A is not a Cauchy matrix.

If $s \in \mathbb{C}^m, t \in \mathbb{C}^n$ are Cauchy points, then

$$C(s, t) = C(s + \alpha 1_m, t + \alpha 1_n)$$

for all $\alpha \in \mathbb{C}$. Consequently, the Cauchy points s, t of a Cauchy matrix $A = [a_{ij}]$ are not uniquely determined by the values a_{ij} . We will show next that this global translation of the Cauchy points is the only source of ambiguity.

Theorem 2.2. The matrix U in (6) satisfies $\ker(U) = \text{span}\{1_{m+n}\}$. Thus, if $\begin{bmatrix} s \\ t \end{bmatrix}$ is a solution of (5)–(6), then the set of all solutions is given by

$$\left\{ \begin{bmatrix} s \\ t \end{bmatrix} + \alpha 1_{m+n} \mid \alpha \in \mathbb{C} \right\}.$$

Proof. Since $U 1_{m+n} = 0$ we have $\text{span}\{1_{m+n}\} \subseteq \ker(U)$. If $z = \begin{bmatrix} x \\ y \end{bmatrix} \in \ker(U)$ with $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$, then

$$x_j 1_n = y, \quad j = 1, \dots, m.$$

In particular, $y = x_1 1_n$, which implies $x_j = x_1$ for $j = 2, \dots, m$, so that $\begin{bmatrix} x \\ y \end{bmatrix} = x_1 1_{m+n}$, giving that $\ker(U) \subseteq \text{span}\{1_{m+n}\}$. \square

In order to remove the ambiguity about the possible Cauchy points that define a given Cauchy matrix we introduce the following definition.

Definition 2.3. Let $A \in \mathbb{C}^{m,n}$ be a Cauchy matrix. We say that $\tilde{s} \in \mathbb{C}^m, \tilde{t} \in \mathbb{C}^n$ are normalized Cauchy points for A , if $A = C(\tilde{s}, \tilde{t})$ and $\left\| \begin{bmatrix} \tilde{s} \\ \tilde{t} \end{bmatrix} \right\|_2$ is minimal among all possible Cauchy points $s \in \mathbb{C}^m, t \in \mathbb{C}^n$ with $A = C(s, t)$.

Algorithm 1 Optimal recovery of normalized Cauchy points

Input: Cauchy matrix $A = [a_{ij}] \in \mathbb{C}^{m,n}$. (Thus, $a_{ij} \neq 0$ for all i, j .)

Output: Normalized Cauchy points \tilde{s}, \tilde{t} such that $A = C(\tilde{s}, \tilde{t})$.

- 1: $s(1) \leftarrow 0$ {Choice arbitrary}
 - 2: $t(1:n) \leftarrow s(1) - A(1, 1:n)^{[-1]}$
 - 3: $s(2:m) \leftarrow t(1) + A(2:m, 1)^{[-1]}$
 - 4: $\alpha_* \leftarrow \frac{1}{m+n} (\sum s_i + \sum t_j)$
 - 5: $\tilde{s} \leftarrow s - \alpha_* \mathbf{1}_m$
 - 6: $\tilde{t} \leftarrow t - \alpha_* \mathbf{1}_n$
-

If $A = C(s, t)$, then normalized Cauchy points for A can be found by solving the minimization problem

$$\min_{\alpha \in \mathbb{C}} \left\| \begin{bmatrix} s \\ t \end{bmatrix} - \alpha \mathbf{1}_{m+n} \right\|_2^2.$$

The unique solution is given by

$$\alpha_* := \frac{\mathbf{1}_{m+n}^T \begin{bmatrix} s \\ t \end{bmatrix}}{\mathbf{1}_{m+n}^T \mathbf{1}_{m+n}} = \frac{\mathbf{1}_{m+n}^T \begin{bmatrix} s \\ t \end{bmatrix}}{m+n},$$

and hence \tilde{s}, \tilde{t} with

$$\begin{bmatrix} \tilde{s} \\ \tilde{t} \end{bmatrix} := \begin{bmatrix} s \\ t \end{bmatrix} - \alpha_* \mathbf{1}_{m+n}$$

are normalized Cauchy points for A .

As described above, if A is a Cauchy matrix, then Cauchy points for A can be computed by solving the system (5)–(6). Since the matrix U has rank $m+n-1$ (cf. Theorem 2.2), the points can be computed by solving any full-rank subsystem of (5)–(6) with $m+n-1$ rows. Due to the simple structure of U , the solution of this subsystem can be computed in $\mathcal{O}(m+n)$ operations. One possible algorithm is shown in Algorithm 1. At the end of the algorithm we normalize the computed Cauchy points (according to Definition 2.3), which can be achieved in $\mathcal{O}(m+n)$ operations as well. Note that only the first row and column of A are accessed by the algorithm.

If we do not know whether A is a Cauchy matrix, we can still apply Algorithm 1 to A . Since the algorithm only considers the first row and column of A , it then costs (at most) mn operations to check whether indeed $A = C(\tilde{s}, \tilde{t})$.

We summarize these observations in the following result.

Theorem 2.4. *If $A \in \mathbb{C}^{m,n}$ is a Cauchy matrix, then Algorithm 1 yields normalized Cauchy points $\tilde{s} \in \mathbb{C}^m$, $\tilde{t} \in \mathbb{C}^n$ with $A = C(\tilde{s}, \tilde{t})$ in $\mathcal{O}(m+n)$ operations. Moreover, for any matrix $A \in \mathbb{C}^{m,n}$ it can be decided in $\mathcal{O}(mn)$ operations whether A is a Cauchy matrix.*

Note that neither the recovery of Cauchy points, nor recognizing Cauchy structure can be achieved asymptotically faster than stated in this theorem.

3 Approximation with Cauchy matrices

In order to (best) approximate a given matrix $A \in \mathbb{C}^{m,n}$ (having only nonzero entries) by a Cauchy matrix, we would ideally like to solve the nonlinear optimization problem (3). As described in the Introduction, we will instead solve the *linearization* of this problem given by (4). Using the notation of Section 2, this standard linear least squares problem can be equivalently written as (cf. (5)–(6))

$$\min_{s,t} \|U \begin{bmatrix} s \\ t \end{bmatrix} - b\|_2^2. \quad (7)$$

Algorithm 1 from Section 2 is clearly inappropriate in this context, as there is no guarantee that the submatrix of U picked for the reconstruction of the Cauchy points yields any useful *global* approximation of the given data when A is not a Cauchy matrix. Our main goal in Section 3.1 is to derive an algorithm of optimal complexity $\mathcal{O}(mn)$ for solving (7). In Section 3.2 we relate the (optimal) solution obtained by this algorithm to the original problem (3).

3.1 Fast solution of the least squares problem

We will solve the least squares problem (7) using the singular value decomposition of the matrix U . We have already characterized the kernel of U in Theorem 2.2. The following result gives a complete characterization of the nonzero singular values and corresponding singular vectors.

Lemma 3.1. *The nonzero singular values of the matrix U in (6) are*

$$\begin{aligned} \sqrt{m+n} & \quad (\text{of multiplicity one}), \\ \sqrt{m} & \quad (\text{of multiplicity } n-1), \\ \sqrt{n} & \quad (\text{of multiplicity } m-1). \end{aligned}$$

Moreover, the corresponding right singular vectors can be characterized as

$$\begin{aligned} \sqrt{m+n} : & \quad \text{span} \left\{ \begin{bmatrix} \sqrt{\frac{n}{m}} 1_m \\ -\sqrt{\frac{m}{n}} 1_n \end{bmatrix} \right\}, \\ \sqrt{m} : & \quad \text{span} \left\{ \begin{bmatrix} 0_m \\ v \end{bmatrix} \mid v \in \mathbb{C}^n, 1_n^T v = 0 \right\}, \\ \sqrt{n} : & \quad \text{span} \left\{ \begin{bmatrix} v \\ 0_n \end{bmatrix} \mid v \in \mathbb{C}^m, 1_m^T v = 0 \right\}, \end{aligned}$$

and the corresponding left singular vectors can be characterized as

$$\begin{aligned} \sqrt{m+n} : & \quad \text{span} \{1_{mn}\}, \\ \sqrt{m} : & \quad \text{span} \{1_m \otimes v \mid v \in \mathbb{C}^n, 1_n^T v = 0\}, \\ \sqrt{n} : & \quad \text{span} \{v \otimes 1_n \mid v \in \mathbb{C}^m, 1_m^T v = 0\}. \end{aligned}$$

Proof. The claims can be verified by straightforward computations using the matrix

$$U^T U = \begin{bmatrix} nI_m & -1_{m,n} \\ -1_{n,m} & mI_n \end{bmatrix}$$

for the right singular vectors, and the matrix

$$U U^T = I_m \otimes 1_{n,n} + 1_{m,m} \otimes I_n$$

for the left singular vectors. \square

The next theorem gives an explicit formula for the solution of (7), which in particular shows that this solution can be computed fast. We denote the Moore-Penrose pseudoinverse of U by U^+ .

Theorem 3.2. *Let $A \in \mathbb{C}^{m,n}$ have only nonzero entries. Let $b := \text{vec}(A^{[-T]})$ and*

$$r := \frac{1}{n} A^{[-1]} 1_n \in \mathbb{C}^m, \quad c := \frac{1}{m} A^{[-T]} 1_m \in \mathbb{C}^n, \quad \sigma := \frac{1}{mn} 1_m^T A^{[-1]} 1_n.$$

Then the minimum norm solution of $\min_{s,t} \|U \begin{bmatrix} s \\ t \end{bmatrix} - b\|_2$ has the form

$$U^+ b = \begin{bmatrix} r - \frac{m\sigma}{m+n} 1_m \\ -c + \frac{n\sigma}{m+n} 1_n \end{bmatrix}, \quad (8)$$

which can be computed in $\mathcal{O}(mn)$ operations. Moreover, $U^+ b$ yields Cauchy points if and only if

$$U U^+ b \neq 0 \quad (\text{componentwise}), \quad (9)$$

or, equivalently,

$$r_i + c_j \neq \sigma \quad \text{for all } i, j. \quad (10)$$

Proof. For an integer $k \geq 1$, we denote by $Q_k \in \mathbb{R}^{k,k-1}$ a matrix whose columns form an orthogonal basis for the linear subspace $\{v \in \mathbb{C}^k \mid 1_k^T v = 0\}$, so that $Q_k^T Q_k = I_{k-1}$ and $1_k^T Q_k = 0$. The characterization of the singular values of U in Lemma 3.1 shows that

$$W = \begin{bmatrix} \sqrt{\frac{n}{m(m+n)}} 1_m & 0_{m,n-1} & Q_m & \frac{1}{\sqrt{m+n}} 1_m \\ -\sqrt{\frac{m}{n(m+n)}} 1_n & Q_n & 0_{n,m-1} & \frac{1}{\sqrt{m+n}} 1_n \end{bmatrix} \in \mathbb{R}^{m+n,m+n} \quad (11)$$

is orthogonal, and yields a diagonalization $U^T U = W \Lambda W^T$ with

$$\Lambda = \text{diag}(m+n, \underbrace{m, \dots, m}_{n-1}, \underbrace{n, \dots, n}_{m-1}, 0),$$

so that

$$U^+ = W \Lambda^+ W^T U^T. \quad (12)$$

Algorithm 2 Minimum 2-norm solution of the least squares problem (7).

Input: Matrix $A = [a_{ij}] \in \mathbb{C}^{m,n}$ with $a_{ij} \neq 0$ for all i, j .

Output: $\begin{bmatrix} s \\ t \end{bmatrix} = U^+ \text{vec}(A^{[-T]})$.

- 1: $r \leftarrow \frac{1}{n} A^{[-1]} \mathbf{1}_n$
 - 2: $c \leftarrow \frac{1}{m} A^{[-T]} \mathbf{1}_m$
 - 3: $\sigma \leftarrow \frac{1}{mn} \mathbf{1}_m^T A^{[-1]} \mathbf{1}_n$
 - 4: $s \leftarrow r - \frac{m\sigma}{m+n} \mathbf{1}_m$
 - 5: $t \leftarrow \frac{n\sigma}{m+n} \mathbf{1}_n - c$ {min. 2-norm solution is automatically normalized}
-

Since $\mathbf{1}_m^T Q_m = 0$, the matrix $\hat{Q} = [Q_m, m^{-\frac{1}{2}} \mathbf{1}_m]$ is orthogonal and hence $I_m = \hat{Q} \hat{Q}^T = Q_m Q_m^T + \frac{1}{m} \mathbf{1}_{m,m}$, which implies that

$$Q_m Q_m^T = I_m - \frac{1}{m} \mathbf{1}_{m,m}. \quad (13)$$

Noting that $U^T b = \begin{bmatrix} nr \\ -mc \end{bmatrix}$, we compute from (12), using (13),

$$\begin{aligned} U^+ b &= W \Lambda^+ W^T \begin{bmatrix} nr \\ -mc \end{bmatrix} = W \Lambda^+ \begin{bmatrix} \sqrt{mn(m+n)}\sigma \\ -mQ_n^T c \\ nQ_m^T r \\ 0 \end{bmatrix} = W \begin{bmatrix} \frac{\sqrt{mn}\sigma}{\sqrt{m+n}} \\ -Q_n^T c \\ Q_m^T r \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} r - \frac{m\sigma}{m+n} \mathbf{1}_m \\ -c + \frac{n\sigma}{m+n} \mathbf{1}_n \end{bmatrix}. \end{aligned}$$

Evaluating the last expression for $U^+ b$ takes $\mathcal{O}(mn)$ operations.

Finally, with $\begin{bmatrix} s \\ t \end{bmatrix} := U^+ b$ the condition (9) simply means that $s_i - t_j \neq 0$ for all i, j , or, equivalently,

$$\left(r_i - \frac{m\sigma}{m+n} \right) - \left(-c_j + \frac{n\sigma}{m+n} \right) = r_i + c_j - \sigma \neq 0$$

for all i, j . □

Note that r and c in Theorem 3.2 are the vectors of row and column means of the matrix $A^{[-1]}$, respectively, while σ is the mean of all its entries. Moreover, for a Cauchy matrix $A = C(s, t)$ the condition (10) reduces to $s_i - t_j \neq 0$ for all i, j .

The overall algorithm for computing $U^+ b$ according to Theorem 3.2 is shown in Algorithm 2.

Remark 3.3. An explicitly constructed matrix Q_m satisfying the requirements in the proof of Theorem 3.2 is given in A. Consequently, a singular value decomposition of the matrix U , based on Lemma 3.1, can be constructed explicitly.

The following example gives a matrix A with only nonzero entries for which Algorithm 2 does not yield Cauchy points.

Example 3.4. Let $0 \neq \alpha \in \mathbb{C}$ and consider the matrix

$$A = \begin{bmatrix} \frac{1}{\alpha} & -\frac{1}{\alpha+2} \\ -\frac{1}{\alpha-2} & \frac{1}{\alpha} \end{bmatrix} \quad \text{so that} \quad A^{[-T]} = \begin{bmatrix} \alpha & -(\alpha-2) \\ -(\alpha+2) & \alpha \end{bmatrix},$$

which gives $r = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\sigma = 0$. The condition (10) does not hold, so that U^+b in (8), or the output of Algorithm 2 applied to A , does not give Cauchy points.

In the next section we will derive conditions under which the output of Algorithm 2 results in good approximations to the original problem (3).

3.2 Approximation bounds

For each matrix $A \in \mathbb{C}^{m,n}$ with only nonzero entries a minimum 2-norm solution $\hat{z} = \begin{bmatrix} \hat{s} \\ \hat{t} \end{bmatrix}$ of the least squares problem (7) and hence of (4) can be computed in $\mathcal{O}(mn)$ operations using Algorithm 2. Of course, without further assumptions we cannot expect that \hat{z} closely approximates the solution of the nonlinear problem (3). Below we will derive a bound on $\|A - C(\hat{s}, \hat{t})\|_F$, and we will bound $\|\begin{bmatrix} \hat{s} \\ \hat{t} \end{bmatrix} - \begin{bmatrix} s \\ t \end{bmatrix}\|_2$ for a perturbed Cauchy matrix $A = C(s, t) + N$. In our derivations we will use that the Hadamard product is submultiplicative with respect to the Frobenius norm, i.e.,

$$\|A \odot B\|_F \leq \|A\|_F \|B\|_F;$$

see, e.g., [8, equation (3.3.5)].

Our first result connects the residuals of (3) and (4). It shows that if for given vectors s, t the relative residual of the linearization (4) is reasonably small, then s, t are Cauchy points, and their relative error with respect to the original problem (3) is small as well. Note that the theorem applies in particular to the output of Algorithm 2, since it computes an *optimal* solution for the linearization (4). Recall that $D(s, t) = [s_i - t_j] \in \mathbb{C}^{m,n}$.

Theorem 3.5. Let $A \in \mathbb{C}^{m,n}$ have only nonzero entries and let $s \in \mathbb{C}^m$, $t \in \mathbb{C}^n$. Define the residual matrix corresponding to (4) by $R := A^{[-1]} - D(s, t)$. If

$$\|A \odot R\|_M =: \beta < 1, \tag{14}$$

then

$$\min_{i,j} |s_i - t_j| \geq \|A\|_M^{-1} (1 - \beta),$$

and hence, in particular, s, t are Cauchy points. Moreover,

$$\frac{\|A - C(s, t)\|_F}{\|A\|_F} \leq \frac{\beta}{1 - \beta}. \tag{15}$$

Proof. Let $R = [r_{ij}]$, then for all i, j we get

$$|s_i - t_j| = \left| \frac{1}{a_{ij}} - r_{ij} \right| = \frac{|1 - a_{ij}r_{ij}|}{|a_{ij}|} \geq \|A\|_M^{-1} (1 - \beta),$$

which shows the lower bound on $\min_{i,j} |s_i - t_j|$.

In order to prove (15) we compute

$$a_{ij} - \frac{1}{s_i - t_j} = a_{ij} - \frac{1}{\frac{1}{a_{ij}} - r_{ij}} = a_{ij} \left(1 - \frac{1}{1 - a_{ij}r_{ij}} \right) = a_{ij} \frac{a_{ij}r_{ij}}{1 - a_{ij}r_{ij}},$$

so that

$$\left| a_{ij} - \frac{1}{s_i - t_j} \right| \leq |a_{ij}| \frac{\beta}{1 - \beta},$$

giving $\|A - C(s, t)\|_F \leq \frac{\beta}{1 - \beta} \|A\|_F$. \square

The condition (14) can be written as

$$\max_{i,j} \left| \frac{(s_i - t_j) - a_{ij}^{-1}}{a_{ij}^{-1}} \right| = \beta < 1. \quad (16)$$

In words, the maximal componentwise relative error in the linear equations (2) that is made by the vectors s, t has to be smaller than one. This appears to be a natural and in fact minimal assumption on the output of Algorithm 2 so that it gives any useful information about the optimization problems (3) and (4). This maximal componentwise relative error can be larger than the global relative error $\|D(s, t) - A^{[-1]}\|_F / \|A^{[-1]}\|_F$, especially if the entries of A vary greatly in magnitude. In that case the bound (15) (and the approximation error) is adversely affected; see Section 3.3 for an example.

In the next result we investigate how closely the output of Algorithm 2 approximates the Cauchy points of a perturbed Cauchy matrix A .

Theorem 3.6. *Let $A = C(\tilde{s}, \tilde{t}) + N \in \mathbb{C}^{m,n}$, where \tilde{s}, \tilde{t} are normalized Cauchy points, have only nonzero entries. Let $s \in \mathbb{C}^m, t \in \mathbb{C}^n$ be a minimum 2-norm solution of the least squares problem (7), i.e., the output of Algorithm 2 applied to A . If*

$$\|D(\tilde{s}, \tilde{t}) \odot N\|_M =: \gamma < 1, \quad (17)$$

then

$$\frac{\|[\tilde{s}] - [s]\|_2}{\|[\tilde{s}]\|_2} \leq \frac{\sqrt{m+n}}{\min\{\sqrt{m}, \sqrt{n}\}} \frac{\gamma}{1 - \gamma}. \quad (18)$$

Proof. Let us denote $C = C(\tilde{s}, \tilde{t})$, $N = [n_{ij}]$ and define

$$B = [b_{ij}] := A^{[-T]} - C^{[-T]}.$$

Since s, t is a minimum 2-norm least squares solution (cf. Theorem 3.2), we have

$$\begin{bmatrix} s \\ t \end{bmatrix} = U^+ \text{vec}(A^{[-T]}) = U^+ (\text{vec}(C^{[-T]}) + \text{vec}(B)) = \begin{bmatrix} \tilde{s} \\ \tilde{t} \end{bmatrix} + U^+ \text{vec}(B).$$

We thus get

$$\begin{aligned} \left\| \begin{bmatrix} \tilde{s} \\ \tilde{t} \end{bmatrix} - \begin{bmatrix} s \\ t \end{bmatrix} \right\|_2 &= \|U^+ \text{vec}(B)\|_2 \leq \|U^+\|_2 \|B\|_F \\ &= \frac{1}{\min\{\sqrt{m}, \sqrt{n}\}} \|B\|_F, \end{aligned}$$

where we used Lemma 3.1 in the last step.

It remains to bound $\|B\|_F$. Note first that for all i, j we have

$$\begin{aligned} |b_{ji}| &= \left| \left(\frac{1}{\tilde{s}_i - \tilde{t}_j} + n_{ij} \right)^{-1} - \left(\frac{1}{\tilde{s}_i - \tilde{t}_j} \right)^{-1} \right| = \left| (\tilde{s}_i - \tilde{t}_j) \frac{(\tilde{s}_i - \tilde{t}_j)n_{ij}}{1 + (\tilde{s}_i - \tilde{t}_j)n_{ij}} \right| \\ &\leq |(\tilde{s}_i - \tilde{t}_j)| \frac{\gamma}{1 - \gamma}, \end{aligned}$$

resulting in

$$\begin{aligned} \|B\|_F &\leq \|D(\tilde{s}, \tilde{t})\|_F \frac{\gamma}{1 - \gamma} = \|U \begin{bmatrix} \tilde{s} \\ \tilde{t} \end{bmatrix}\|_2 \frac{\gamma}{1 - \gamma} \leq \|U\|_2 \left\| \begin{bmatrix} \tilde{s} \\ \tilde{t} \end{bmatrix} \right\|_2 \frac{\gamma}{1 - \gamma} \\ &= \sqrt{m + n} \left\| \begin{bmatrix} \tilde{s} \\ \tilde{t} \end{bmatrix} \right\|_2 \frac{\gamma}{1 - \gamma}, \end{aligned}$$

where we have again used Lemma 3.1 in the last step. □

The condition (17), i.e.,

$$\left\| [(\tilde{s}_i - \tilde{t}_j)n_{ij}] \right\|_M = \left\| \begin{bmatrix} n_{ij} \\ \frac{1}{\tilde{s}_i - \tilde{t}_j} \end{bmatrix} \right\|_M < 1,$$

ensures that the maximum (component wise) relative noise level is reasonably small. Note also that the constant on the right hand side of (18) is equal to $\sqrt{2}$ when $m = n$.

The two bounds presented in Theorems 3.5 and 3.6 are complementary: On the one hand, a small residual (15) *does not* imply that Algorithm 2 recovers nearby Cauchy points of a noisy Cauchy matrix as in (18). On the other hand, if Algorithm 2 recovers nearby Cauchy points of a noisy Cauchy matrix as in (18), then this *does not* imply that the residual (15) is small. Numerical examples demonstrating this are given in Section 3.3.

Remark 3.7. *Without further assumptions on N it is not guaranteed that the output of Algorithm 2 applied to a noisy Cauchy matrix $A = C(\tilde{s}, \tilde{t}) + N$ (with only nonzero entries) yields Cauchy points. However, considering (9), the output $U^+ \text{vec}(A^{[-T]})$ are indeed Cauchy points if $\|N\|$ is sufficiently small, since the function*

$$\{A \in \mathbb{C}^{m,n} \mid a_{ij} \neq 0\} \rightarrow \mathbb{C}^{mn}, \quad A \mapsto UU^+ \text{vec}(A^{[-T]}),$$

is continuous, and $UU^+ \text{vec}(C(\tilde{s}, \tilde{t})^{[-T]}) \neq 0$ (componentwise). We did not attempt to derive a quantitative bound on N such that $U^+ \text{vec}(A^{[-T]})$ are guaranteed to be Cauchy points; see, however, the conditions (10) and (14).

3.3 Numerical examples

Approximation quality of Algorithm 2

We consider the vectors $s \in \mathbb{C}^{200}$ $t \in \mathbb{C}^{100}$, where the real part consists of equally spaced points in the interval $[-1, 1]$, and imaginary parts set to i and $-i$, respectively, i.e.,

$$s = \text{linspace}(-1, 1, 200) + i, \quad t = \text{linspace}(-1, 1, 100) - i \quad (19)$$

in MATLAB syntax. Consequently, the all the entries of the Cauchy matrix $C := C(s, t)$ have the same magnitude.

In order to study the approximation quality of Algorithm 2, we perturb C by some noise matrix N_δ for a series of increasing noise levels $\delta \in [10^{-16}, 1]$. We consider a random matrix $N \in \mathbb{C}^{200, 100}$ (generated by MATLAB's `randn` function for its real and imaginary parts) and set

$$A_\delta := C + N_\delta, \quad \text{where} \quad N_\delta := \delta * (N \odot |N|^{-1}) \odot |C|. \quad (20)$$

Thus, the relative perturbation of C by N_δ in each component is exactly δ (compare (17)). We apply Algorithm 2 to each such matrix A_δ , and we denote the output by $\hat{z} := \begin{bmatrix} \hat{s} \\ \hat{t} \end{bmatrix} := U^+ \text{vec}(A_\delta^{[-T]})$.

Figure 1 (top) shows, for each noise level δ , the relative approximation error $\frac{\|A_\delta - C(\hat{s}, \hat{t})\|_F}{\|A_\delta\|_F}$. We also plot the bound (15) and the relative error $\frac{\|A_\delta - C(s, t)\|_F}{\|A_\delta\|_F}$ made by the original Cauchy points. We observe that the output \hat{s}, \hat{t} of Algorithm 2 yields an approximation of the given data matrix A_δ by a Cauchy matrix $C(\hat{s}, \hat{t})$ with approximation error linear in the noise level, and that this approximation quality is on par with the original Cauchy points. Moreover, the bound (15) matches the true residual rather well.

The computed Cauchy points \hat{s}, \hat{t} are, however, different from the original ones. Figure 1 (bottom) shows the relative recovery error $\|z - \hat{z}\|_2 / \|z\|_2$, where $z := \begin{bmatrix} s \\ t \end{bmatrix}$. As for the data approximation error, the recovery error behaves linearly in the noise level.

We now study the effect of increasing the range of magnitudes in the coefficients of the Cauchy matrix $C(s, t)$ by setting the imaginary parts of the vectors s and t to $10^{-6}i$ and $-10^{-6}i$ (instead of i and $-i$), respectively, i.e.,

$$s = \text{linspace}(-1, 1, 200) + 10^{-6}i, \quad t = \text{linspace}(-1, 1, 100) - 10^{-6}i. \quad (21)$$

Figure 2 (top) shows that this change leads to an increase of the approximation error $\|A_\delta - C(\hat{s}, \hat{t})\|_F / \|A_\delta\|_F$ by about six orders of magnitude, while the global approximation error of the linearization behaves nicely with respect to the noise level; cf. (16) and corresponding discussion. On the other hand, the relative error of the recovered Cauchy points $\|z - \hat{z}\|_2 / \|z\|_2$ is largely unaffected by this change; see Figure 2 (bottom).

Notice also the “wiggly” behaviour of the blue and red line in Figure 2 (top); this is due to roundoff error in computing the row and column means in Algorithm 2. Using a multiply compensated summation [12] would yield a more stable behaviour (at a $\log(mn)$ factor higher operation count).

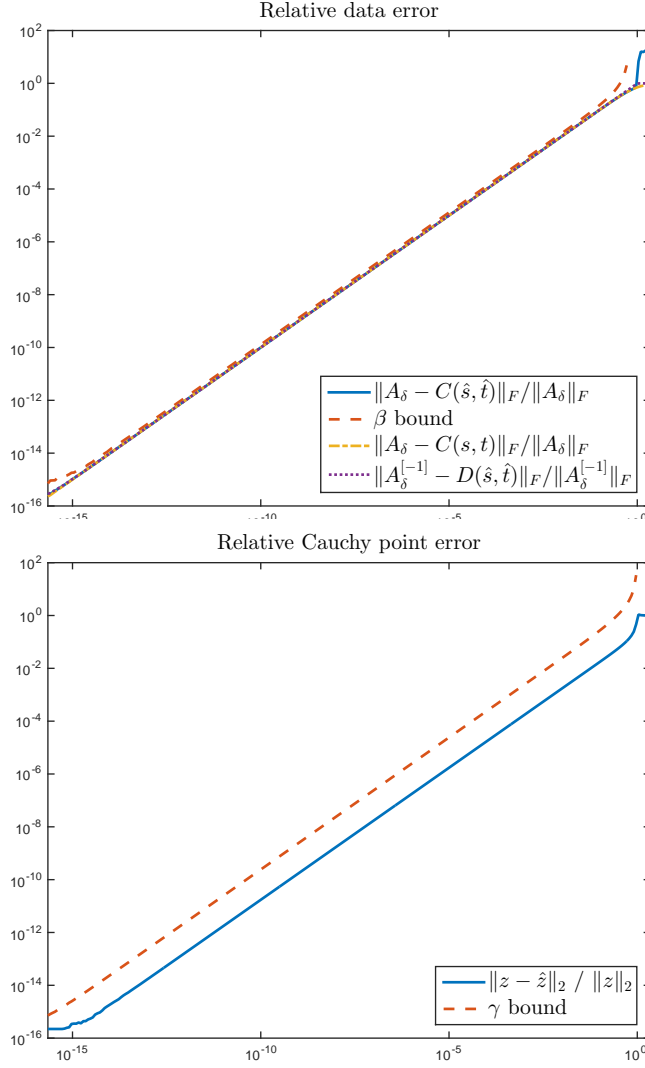


Figure 1: Approximation quality of Algorithm 2 for the data (19)–(20). *Top picture*: Relative data approximation error corresponding to the Cauchy points obtained by Algorithm 2 (solid blue line), the original Cauchy points (dash-dotted yellow line), the bound (15) (dashed red line), and the relative residual of the linearized problem (4) (dotted purple line). All four lines are visually almost indistinguishable. *Bottom picture*: Relative Cauchy points approximation error corresponding to the Cauchy points obtained by Algorithm 2 (solid blue line), and the bound (18) (dashed red line).

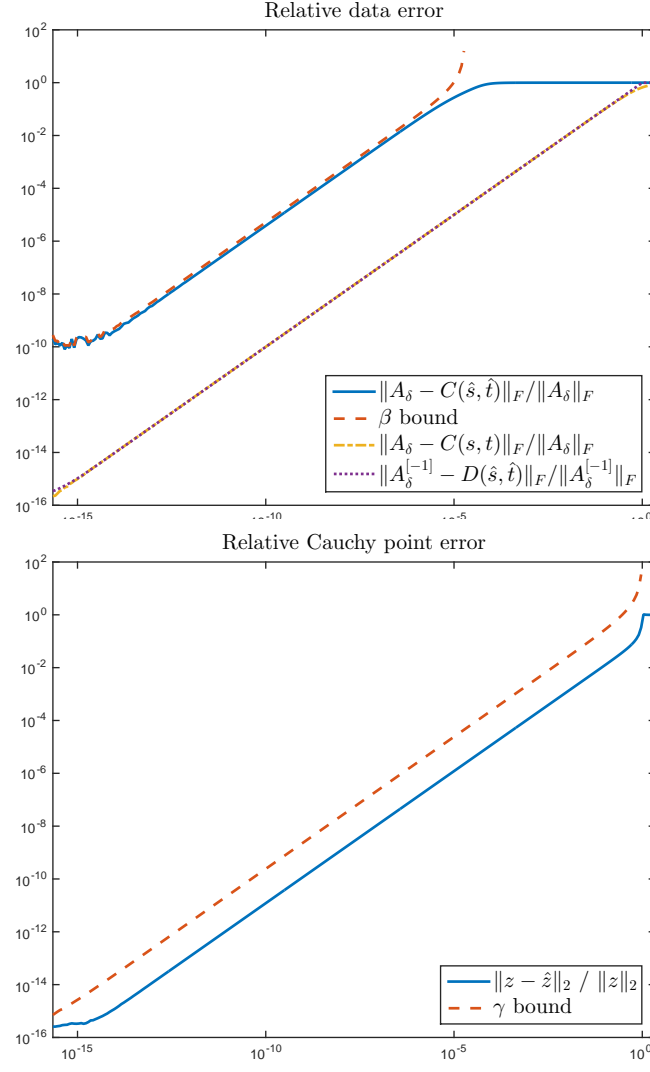


Figure 2: Approximation quality of Algorithm 2 for the data (20)–(21). Notation as in Figure 1.

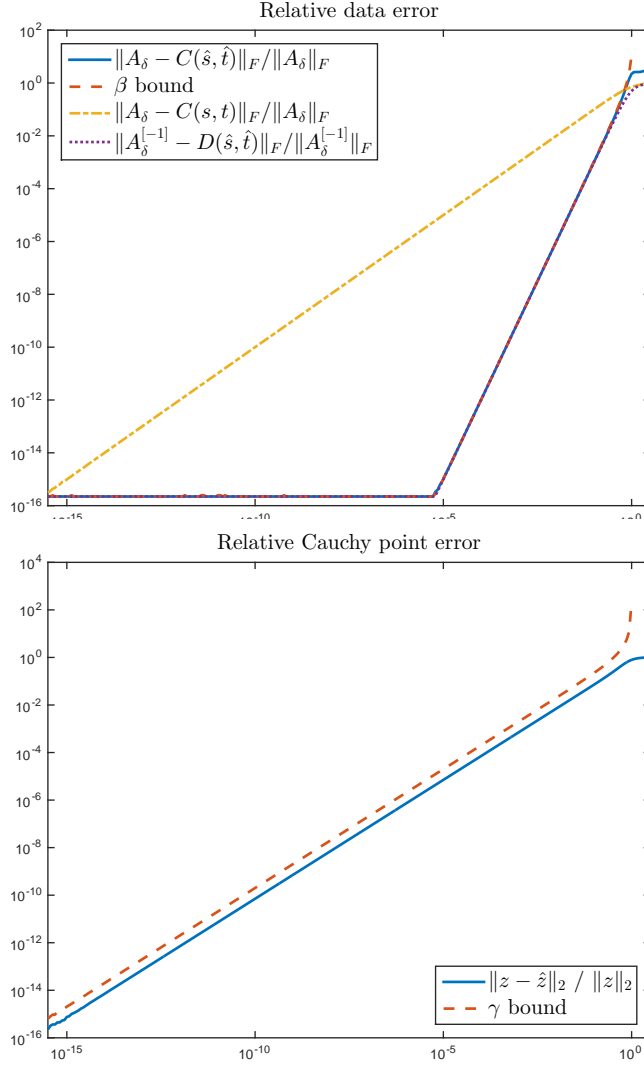


Figure 3: Relative data approximation error (top) and Cauchy point error (bottom) for data $A_\delta = C([1; -1], [i; -i]) + \delta[1, -1; -1, 1]$. (Values smaller than the machine precision ϵ have been set to ϵ for cleaner presentation.)

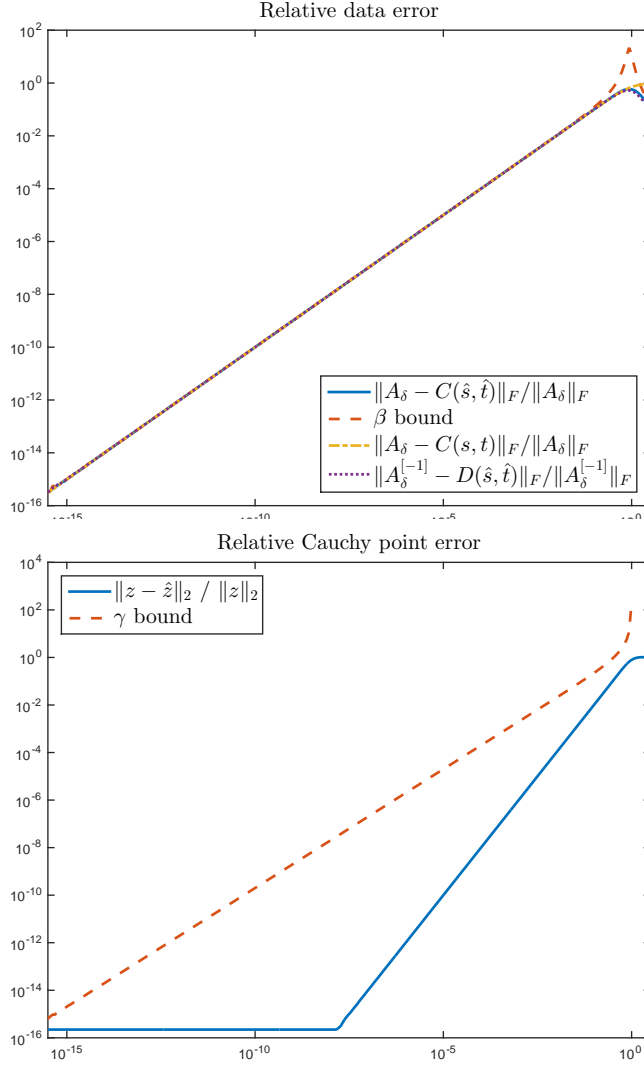


Figure 4: Relative data approximation error (top) and Cauchy point error (bottom) for data $A_\delta = C([1; -1], [i; -i]) + \delta[-1, -1; -1, -1]$. (Values smaller than the machine precision ϵ have been set to ϵ for cleaner presentation.)

Complementarity of the bounds in Theorems 3.5 and 3.6

We consider the 2×2 Cauchy matrix $C(s, t)$ having the (normalized) Cauchy points

$$s = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} i \\ -i \end{bmatrix}.$$

Figure 3 shows the same quantities as in the previous examples for the matrices

$$A_\delta = C(s, t) + \delta \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where δ ranges from 10^{-16} to 1.0. While the error made in the recovered Cauchy points increases linearly in δ , the data approximation residual $\frac{\|A_\delta - C(\hat{s}, \hat{t})\|_F}{\|A_\delta\|_F}$ remains on the machine precision level until $\delta \approx 10^{-5}$. A computation shows that for this particular choice of s, t and N , the residual of the linearization corresponding to the solution $\begin{bmatrix} \hat{s} \\ \hat{t} \end{bmatrix} = U^+ \text{vec}(A_\delta^{[-T]})$ has the form

$$R = A_\delta^{[-1]} - D(\hat{s}, \hat{t}) = \begin{bmatrix} \frac{4\delta^3}{1+4\delta^4} & \frac{-4\delta^3}{1+4\delta^4} \\ \frac{-4\delta^3}{1+4\delta^4} & \frac{4\delta^3}{1+4\delta^4} \end{bmatrix},$$

so that $\beta = \|A_\delta \odot R\|_M$ (see (14)) is smaller than ϵ until $\delta \approx 10^{-5}$. Consequently, the bound (15) implies that the data approximation residual is about the same size.

More generally, when for a Cauchy matrix $C(s, t)$ a perturbation N is such that

$$\text{vec}((C(s, t) + N)^{[-T]}) \in \text{im}(U)$$

(see (6)), the data approximation residual will be zero, while the distance of $\begin{bmatrix} \hat{s} \\ \hat{t} \end{bmatrix} = U^+ \text{vec}(A_\delta^{[-T]})$ to the original Cauchy points $\begin{bmatrix} s \\ t \end{bmatrix}$ can become arbitrarily large.

Using the same Cauchy points as above we now consider a perturbation of the form

$$A_\delta = C(s, t) - \delta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The resulting errors are shown in Figure 4. Now the data approximation error behaves linearly in δ , but the Cauchy points s, t are exactly recovered up to $\delta \approx 10^{-8}$. A computation shows that the output of Algorithm 2 applied to A_δ is

$$\begin{bmatrix} \hat{s} \\ \hat{t} \end{bmatrix} = U^+ \text{vec}(A_\delta^{[-T]}) = \frac{1}{1+4\delta^4} \begin{bmatrix} 1 - 2\delta^2 - 2\delta^3 \\ -1 + 2\delta^2 - 2\delta^3 \\ i + 2i\delta^2 + 2\delta^3 \\ -i - 2i\delta^2 + 2\delta^3 \end{bmatrix},$$

so that, numerically, the recovered Cauchy points are the original ones until $\delta^2 \approx \epsilon$.

More generally, for a Cauchy matrix $C(s, t)$ a perturbation N is such that

$$\text{vec}(C(s, t)^{[-T]} - (C(s, t) + N)^{[-T]}) \in \text{im}(U)^\perp,$$

then Algorithm 2 will recover $\begin{bmatrix} s \\ t \end{bmatrix}$ exactly, while the data approximation error can become arbitrarily large.

4 Concluding remarks

We presented an efficient algorithm for the approximation of a given matrix with a Cauchy matrix. Our approach for solving the approximation problem is based on the solution of a linear least squares problem based on the explicit construction of the pseudoinverse of a structured matrix. It would be very interesting to investigate whether similar approximation algorithms can be derived for other displacement structured matrices like generalized Cauchy matrices or Cauchy-like matrices; see, e.g., [1, 11, 7].

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A An explicit SVD of U

Lemma A.1. *Let $m > 1$, set $n := m - 1$ and $\nu_j := \sqrt{1 + \frac{1}{j}}$ for $j = 1, \dots, n$. Then the unreduced upper Hessenberg matrix*

$$Q_m := \begin{bmatrix} \nu_1 & 2\nu_2 & \dots & n\nu_n \\ -\nu_1 & 2\nu_2 & \dots & n\nu_n \\ & -\nu_2 & \ddots & \vdots \\ & & \ddots & n\nu_n \\ & & & -\nu_n \end{bmatrix}^{[-1]} \in \mathbb{R}^{m, m-1}$$

satisfies $Q_m^T Q_m = I_{m-1}$ and $1_m^T Q_m = 0$. In particular, the columns of Q_m form an orthogonal basis for the subspace $\{v \in \mathbb{C}^m \mid 1_m^T v = 0\}$.

Proof. Let q_i, q_j be the i th and j th column of Q , respectively, and assume without loss of generality that $1 \leq i < j \leq n$. In order to show $Q_m^T Q_m = I_{m-1}$ we compute

$$q_i^T q_j = \sum_{k=1}^i \frac{1}{ij\nu_i\nu_j} - \frac{1}{j\nu_i\nu_j} = \frac{1}{j\nu_i\nu_j} - \frac{1}{j\nu_i\nu_j} = 0,$$

and for $1 \leq j \leq n$,

$$q_j^T q_j = \sum_{k=1}^j \frac{1}{j^2 \nu_j^2} + \frac{1}{\nu_j^2} = \frac{1}{j \nu_j^2} + \frac{1}{\nu_j^2} = \frac{1}{1+j} + \frac{j}{j+1} = 1.$$

The equation $1_m^T Q_m = 0$ follows from $1_m^T q_j = \frac{j}{j \nu_j} - \frac{1}{\nu_j} = 0$.

□

Using the explicitly constructed matrix Q_m in (11), we obtain an explicit matrix of right singular vectors of the matrix U in (6). Orthogonal bases for the eigenspaces of UU^T can also be obtained using Q_m as a building block (cf. the second part of Lemma 3.1), so that a complete SVD of U can be explicitly constructed.

From the special structure of Q_m it is not difficult to see that matrix vector products with Q_m and Q_m^T can be evaluated in $\mathcal{O}(m)$ operations. Consequently, matrix-vector products with the SVD-factors of U can be carried out in constant time per vector component of the output.

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